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Some Relations between Dualities, Polarities, Coupling Functionals, and Conjugations

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0. INTRODUCTION

In this paper we show some relations between dualities, in the sense of Evers and van Maaren [2], polarities, as used, e.g., in Griffin, Aráoz, and Edmonds [4], coupling functionals, in the sense of Moreau [9, 10] and conjugations, as introduced in [15]. The precise definitions of the concepts of duality, polarity, coupling functional, and conjugation will be recalled in the subsequent sections. In the present section we recall some notations and conventions.

If X is any set, 2^X denotes the family of all subsets of X and $\bar{R}^X$ denotes the family of all functionals $f: X \to \bar{R} = R \cup \{-\infty, +\infty\}$, where $R = (-\infty, +\infty)$. If X is a (real) locally convex space, X^* denotes the linear space of all continuous linear functionals $\Phi: X \to R$, endowed with the w^*-topology $\sigma(X^*, X)$.

We shall use on $\bar{R}$ the "upper addition" $\dot{+}$ and the "lower addition" $\underset{\cdot}{+}$ i.e. (see [9, 10]),

$$a \dot{+} b = a \underset{\cdot}{+} b = a + b \qquad \text{if} \quad R \cap \{a, b\} \neq \emptyset \text{ or } a = b = \pm\infty, \tag{0.1}$$

$$a \dot{+} b = +\infty, \quad a \underset{\cdot}{+} b = -\infty \quad \text{if} \quad a = -b = +\infty; \tag{0.2}$$

for the rules of computation with $\dot{+}$ and $\underset{\cdot}{+}$ we shall refer to [10].

We recall that, for any set X, the usual structures (lattice, etc.) of $\bar{R}^X$ are defined pointwise on X (i.e., $(\sup_{i \in I} f_i)(x) = \sup_{i \in I} f_i(x)$ for all $x \in X$, etc.). We shall use the same notation for the elements of $\bar{R}$ and the constant functionals on any set X, with values in $\bar{R}$; thus, if $d \in \bar{R}$, we shall also denote by d the functional $h \in \bar{R}^X$ defined by $h(x) = d$ $(x \in X)$.

We shall adopt the usual conventions $\inf \emptyset = +\infty$, $\sup \emptyset = -\infty$, where $\emptyset$ denotes the empty set, and (see, e.g., [2, p. 9])

$$\bigcap_{A \in \emptyset} A = X, \tag{0.3}$$

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i.e., the intersection of the elements of the empty family of subsets of X is X; this concerns also the formulae in which, for reasons of typographical simplicity, the intersection occurs only implicitly, such as the definitions of polars, hulls, etc.

We recall that the *indicator functional* $\chi_G \in \bar{R}^X$ of a set $G \subset X$ is defined by

$$\begin{aligned} \chi_G(y) &= 0 \qquad &\text{if} \quad y \in G \\ &= +\infty \qquad &\text{if} \quad y \notin G; \end{aligned} \tag{0.4}$$

in the sequel, an important role will be played by the indicator functionals $\chi_{\{x\}}$ of singletons $\{x\}$, where $x \in X$.

Finally, throughout the sequel, unless otherwise stated, X and W will denote two arbitrary non-empty sets.

1. Dualities and Polarities

We recall that, following Evers and van Maaren [2, p. 8], a "*duality* between two sets X and W" is a mapping $\Delta: 2^X \to 2^W$ satisfying, for any index set $I \neq \emptyset$,

$$\Delta\left(\bigcup_{i \in I} A_i\right) = \bigcap_{i \in I} \Delta(A_i) \qquad (\{A_i\}_{i \in I} \subset 2^X); \tag{1.1}$$

in particular,

$$\Delta(A) = \Delta\left(\bigcup_{x \in A} \{x\}\right) = \bigcap_{x \in A} \Delta(\{x\}) \qquad (A \subset X), \tag{1.2}$$

whence, by (0.3), $\Delta(\emptyset) = W$. Numerous examples of dualities, occurring in various branches of mathematics, are given in [2]; see also examples 2.1–2.3 and 4.1 below.

Let us also recall that if X and W are two sets and $\Omega \subset X \times W$, then (see, e.g., Griffin, Aráoz, and Edmonds [4]), the "*Ω-polar*" of any set $A \subset X$ is defined as

$$A^{\rho(\Omega)} = \{w \in W \mid (x, w) \in \Omega \ (x \in A)\} \tag{1.3}$$

and the mapping $\rho(\Omega)$: $2^X \to 2^W$ is called the "*polarity* between subsets of X and subsets of W *defined by* Ω"; in particular,

$$\{x\}^{\rho(\Omega)} = \{w \in W \mid (x, w) \in \Omega\} \qquad (x \in X), \tag{1.4}$$

whence

$$A^{\rho(\Omega)} = \bigcap_{x \in A} \{x\}^{\rho(\Omega)} \qquad (A \subset X), \tag{1.5}$$

and thus, by (0.3), $\varnothing^{\rho(\Omega)} = W$. For some examples of polarities, see Example 2.4 below. Our first result is

THEOREM 1.1. (a) *Every polarity $\rho(\Omega)\colon 2^X \to 2^W$ is a duality.*

(b) *For every duality $\Delta\colon 2^X \to 2^W$ there exists a unique set $\Omega_\Delta \subset X \times W$ such that $\Delta = \rho(\Omega_\Delta)$, namely,*

$$\Omega_\Delta = \{(x, w) \in X \times W \mid w \in \Delta(\{x\})\}. \tag{1.6}$$

Proof. (a) For any $A_i \subset X$ $(i \in I)$, we have

$$\left(\bigcup_{i \in I} A_i\right)^{\rho(\Omega)} = \left\{ w \in W \,\middle|\, (x, w) \in \Omega \left(x \in \bigcup_{i \in I} A_i \right) \right\}$$

$$= \bigcap_{i \in I} \{w \in W \mid (x, w) \in \Omega\ (x \in A_i)\} = \bigcap_{i \in I} A_i^{\rho(\Omega)}.$$

(b) Let $\Delta\colon 2^X \to 2^W$ be a duality and define $\Omega_\Delta \subset X \times W$ by (1.6). Then $(x, w) \in \Omega_\Delta$ if and only if $w \in \Delta(\{x\})$, whence, by (1.4),

$$\Delta(\{x\}) = \{w \in W \mid (x, w) \in \Omega_\Delta\} = \{x\}^{\rho(\Omega_\Delta)} \qquad (x \in X). \tag{1.7}$$

Hence, by (1.2) and (1.5), we obtain

$$\Delta(A) = \bigcap_{x \in A} \Delta(\{x\}) = \bigcap_{x \in A} \{x\}^{\rho(\Omega_\Delta)} = A^{\rho(\Omega_\Delta)} \qquad (A \subset X), \tag{1.8}$$

so $\Delta = \rho(\Omega_\Delta)$. Finally, the set Ω_Δ is unique, since *any polarity $\rho(\Omega)$ determines uniquely the set Ω*; indeed, by (1.4), we have $(x, w) \in \Omega$ if and only if $w \in \{x\}^{\rho(\Omega)}$, that is,

$$\Omega = \{(x, w) \in X \times W \mid w \in \{x\}^{\rho(\Omega)}\}. \tag{1.9}$$

Remark 1.1. (a) Theorem 1.1 shows that a *mapping $\Delta\colon 2^X \to 2^W$ is a duality if and only if it is a polarity.* Moreover, *there exists a one-to-one correspondence between dualities $\Delta\colon 2^X \to 2^W$ and subsets Ω* of $X \times W$: to each duality Δ there corresponds the set Ω_Δ of (1.6) and, conversely, to each $\Omega \subset X \times W$ there corresponds the duality $\Delta = \rho(\Omega)$ of (1.3).

(b) *In the sequel we shall consider only dualities Δ* and then we shall mention briefly how to transpose our results on dualities into corresponding results on polarities $\rho(\Omega)$, or equivalently, on subsets Ω of $X \times W$.

2. Dualities and Coupling Functionals

We recall that if X and W are two sets, then any functional $\varphi: X \times W \to \bar{R}$ is called [9, 10] a *coupling functional.*

Definition 2.1. For any coupling functional $\varphi: X \times W \to \bar{R}$, we define *the duality* $\Delta_\varphi: 2^X \to 2^W$ *associated to* φ, by

$$\Delta_\varphi(A) = \{w \in W \mid \varphi(x, w) \geqslant -1 \ (x \in A)\} \qquad (A \subset X). \tag{2.1}$$

Remark 2.1. (a) By (0.3), $\Delta_\varphi(\varnothing) = W$. On the other hand, we have $\Delta_\varphi(X) = \varnothing$ if and only if

$$\inf_{x \in X} \varphi(x, w) < -1 \qquad (w \in W), \tag{2.2}$$

which is satisfied, e.g., in Examples 2.2 and 2.3 below.

(b) Δ_φ is indeed a duality, since for any $A_i \subset X$ $(i \in I)$ we have

$$\Delta_\varphi\left(\bigcup_{i \in I} A_i\right) = \left\{w \in W \,\middle|\, \varphi(x, w) \geqslant -1 \left(x \in \bigcup_{i \in I} A_i\right)\right\}$$
$$= \bigcap_{i \in I} \{w \in W \mid \varphi(x, w) \geqslant -1 \ (x \in A_i)\} = \bigcap_{i \in I} \Delta_\varphi(A_i).$$

For example, let us give the dualities associated to some coupling functionals φ considered in [15].

Example 2.1. Let X be a locally convex space, $W = X^*$ and φ_0 the "natural (bilinear) coupling functional," i.e.,

$$\varphi_0(x, \Phi) = \Phi(x) \qquad (x \in X, \Phi \in X^*). \tag{2.3}$$

Then, by (2.1),

$$\Delta_{\varphi_0}(A) = \{\Phi \in X^* \mid \Phi(x) \geqslant -1 \ (x \in A)\} \qquad (A \subset X), \tag{2.4}$$

and hence $-\Delta_{\varphi_0}(A) = \Delta_{-\varphi_0}(A) = A^0$, the usual polar of the set A (see Example 2.4 below).

Remark 2.2. One might change the definition of $\Delta_\varphi(A)$, replacing $\varphi(x, w) \geqslant -1$ by $\varphi(x, w) \leqslant 1$ in (2.1) and then for φ_0 of (2.3) we would have $\Delta_{\varphi_0}(A) = A^0$; however, we need $\Delta_\varphi(A)$ as in (2.1) above, in view of our subsequent results (see Remark 3.2(a)).

EXAMPLE 2.2. Let X be a locally convex space, $W = X^* \times R$ and (see [15, Example 2.3]) let

$$\varphi(x, (\Phi, \lambda)) = -\chi_{\{y \in X \mid \Phi(y) \geqslant \lambda\}}(x) \qquad (x \in X, \Phi \in X^*, \lambda \in R). \tag{2.5}$$

Then, by (2.1),

$$\begin{aligned} \Delta_\varphi(A) &= \{(\Phi, \lambda) \in X^* \times R \mid -\chi_{\{y \in X \mid \Phi(y) \geqslant \lambda\}}(x) \geqslant -1 \ (x \in A)\} \\ &= \{(\Phi, \lambda) \in X^* \times R \mid \inf_{x \in A} \Phi(x) \geqslant \lambda\} \qquad (A \subset X), \end{aligned} \tag{2.6}$$

and hence $-\Delta_\varphi(A) = \{(\Phi, \lambda) \mid \sup \Phi(A) \leqslant \lambda\}$, the "$\varphi_0$-conjugate set to A" in the sense of [13, p. 14, Definition 2.2] (for φ_0 of (2.3)).

EXAMPLE 2.3. Let $X = W$ be an arbitrary set and (see [15, Example 2.4]) let

$$\varphi(x, w) = -\chi_{\{x\}}(w) \qquad (x, w \in X). \tag{2.7}$$

Then, by (2.1), for any $A \subset X$ we have

$$\begin{aligned} \Delta_\varphi(A) &= \{w \in W \mid -\chi_{\{x\}}(w) \geqslant -1 \ (x \in A)\} \\ &= A \qquad \text{if} \quad A \text{ is a singleton,} \\ &= \varnothing \qquad \text{otherwise.} \end{aligned} \tag{2.8}$$

Definition 2.1 associates to each coupling functional $\varphi: X \times W \to \bar{R}$ a duality $\Delta_\varphi: 2^X \to 2^W$. In order to proceed in the opposite direction, let us first give

DEFINITION 2.2. We shall say that a coupling functional $\varphi: X \times W \to \bar{R}$ is *of type* $\{0, -\infty\}$, if $\varphi(X \times W) \subset \{0, -\infty\}$, i.e., if φ can assume only the values 0 and $-\infty$.

Note that φ_0 of Example 2.1 is not of type $\{0, -\infty\}$ (moreover, φ_0 never assumes the value $-\infty$), but the φ's of Examples 2.2 and 2.3 are of type $\{0, -\infty\}$. In the sequel the functionals of type $\{0, -\infty\}$ will play an important role and usually we shall denote such functionals by φ_1. Obviously, for φ_1 of type $\{0, -\infty\}$ and any $c \in R$, $c \leqslant 0$, we have

$$\begin{aligned} \Delta_{\varphi_1}(A) &= \{w \in W \mid \varphi_1(x, w) = 0 \ (x \in A)\} \\ &= \{w \in W \mid \varphi_1(x, w) \geqslant c \ (x \in A)\} \qquad (A \subset X). \end{aligned}$$

THEOREM 2.1. *For each duality Δ there exists a unique coupling functional φ_1 of type $\{0, -\infty\}$, such that $\Delta = \Delta_{\varphi_1}$, namely,*

$$\varphi_1(x, w) = -\chi_{\Delta(\{x\})}(w) \qquad (x \in X, w \in W). \tag{2.9}$$

Proof. By (2.1), (2.9), and (1.2), we have

$$\begin{aligned}\Delta_{\varphi_1}(A) &= \{w \in W \mid -\chi_{\Delta(\{x\})}(w) \geqslant -1 \ (x \in A)\} \\ &= \{w \in W \mid w \in \Delta(\{x\}) \ (x \in A)\} \\ &= \bigcap_{x \in A} \Delta(\{x\}) = \Delta(A) \qquad (A \subset X).\end{aligned}$$

Furthermore, φ_1 is unique, since *for any φ_1 of type $\{0, -\infty\}$, the duality Δ_{φ_1} determines uniquely the coupling functional φ_1 of type $\{0, -\infty\}$*; indeed, by (2.1) we have $\varphi_1(x, w) \geqslant -1$ if and only if $w \in \Delta_{\varphi_1}(\{x\})$, whence, by $\varphi_1(X \times W) \subset \{0, -\infty\}$, there follows

$$\varphi_1(x, w) = -\chi_{\Delta_{\varphi_1}(\{x\})}(w) \qquad (x \in X, w \in W). \tag{2.10}$$

Remark 2.3. (a) By Theorem 2.1, we have a one-to-one correspondence between dualities $\Delta: 2^X \to 2^W$ and coupling functionals φ_1 of type $\{0, -\infty\}$. We shall call φ_1 of (2.9) *the coupling functional of type $\{0, -\infty\}$ associated to the duality Δ* and we shall denote it by $(\varphi_1)_\Delta$.

(b) Part of Theorem 2.1 is similar to a result of Evers and van Maaren, who have stated [2, Sect. 2] that for any coupling functional $\varphi: X \times W \to R$ one can define a duality $\Delta_\varphi^0: 2^X \to 2^W$ by

$$\Delta_\varphi^0(A) = \{w \in W \mid \varphi(x, w) \leqslant 0 \ (x \in A)\} \qquad (A \subset X), \tag{2.11}$$

and that, conversely, any duality Δ "can be represented in this manner," using the coupling functional $\varphi(x, w) = 0$ if $w \in \Delta(\{x\})$ and $\varphi(x, w) = 1$ if $w \notin \Delta(\{x\})$. Moreover, similarly to the above proof of Theorem 2.1, one can show that φ satisfying $\Delta = \Delta_\varphi^0$ is unique, among the φ's "of type $\{0, 1\}$" (this has not been observed in [2]).

(c) Similarly to (b), one can show that Theorem 2.1 remains valid if we replace in it "φ of type $\{0, -\infty\}$" by φ's of various other types (provided that we modify suitably (2.9)), or the duality Δ_φ of (2.1) by various other dualities Δ_φ, or both. Similar remarks can be also made for the other results of this paper, but we shall not mention them again in the sequel (with a few exceptions, such as Example 2.4 and Remark 3.2).

Theorem 2.1 suggests to introduce an equivalence relation in the family of all coupling functionals $\varphi: X \times W \to \bar{R}$, as follows:

DEFINITION 2.3. We shall say that two coupling functionals φ, φ': $X \times W \to \bar{R}$ are *equivalent*, and we shall write $\varphi \sim \varphi'$, if the associated dualities satisfy $\Delta_\varphi = \Delta_{\varphi'}$, i.e., $\Delta_\varphi(A) = \Delta_{\varphi'}(A)$ $(A \subset X)$.

Remark 2.4. We have $\varphi \sim \varphi'$ if and only if

$$\{w \in W \mid \varphi(x, w) \geqslant -1\} = \{w \in W \mid \varphi'(x, w) \geqslant -1\} \qquad (x \in X), \quad (2.12)$$

i.e., $\Delta_\varphi(\{x\}) = \Delta_{\varphi'}(\{x\})$ $(x \in X)$. Indeed, the "only if" part is obvious (taking $A = \{x\}$) and the "if" part follows from (1.2). Clearly, (2.12) holds if and only if

$$\{x \in X \mid \varphi(x, w) \geqslant -1\} = \{x \in X \mid \varphi'(x, w) \geqslant -1\} \qquad (w \in W). \quad (2.12')$$

COROLLARY 2.1. *Each coupling functional φ: $X \times W \to \bar{R}$ is equivalent to a unique coupling functional φ_1 of type $\{0, -\infty\}$, namely,*

$$\begin{aligned} \varphi_1(x, w) &= 0 && \text{if} \quad \varphi(x, w) \geqslant -1, \\ &= -\infty && \text{if} \quad \varphi(x, w) < -1. \end{aligned} \qquad (2.13)$$

Proof. By Definition 2.1, for any coupling functional φ: $X \times W \to \bar{R}$ we have a unique associated duality Δ_φ: $2^X \to 2^W$, given by (2.1). By Theorem 2.1, for this Δ_φ there exists a unique coupling functional φ_1 of type $\{0, -\infty\}$, such that $\Delta_\varphi = \Delta_{\varphi_1}$, given by

$$\begin{aligned} \varphi_1(x, w) &= -\chi_{\Delta_\varphi(\{x\})}(w) \\ &= -\chi_{\{w' \in W \mid \varphi(x, w') \geqslant -1\}}(w) \qquad (x \in X,\ w \in W), \end{aligned}$$

which is nothing else than (2.13). Alternatively, it is obvious that for φ_1 defined by (2.13) we have, by (2.1), $\Delta_{\varphi_1} = \Delta_\varphi$, so $\varphi_1 \sim \varphi$ and, by the proof of Theorem 2.1, Δ_{φ_1} determines uniquely the φ_1 of type $\{0, -\infty\}$.

Remark 2.5. By the above, we have a one-to-one correspondence $\Delta \to [\varphi]$ between the dualities Δ: $2^X \to 2^W$ and the equivalence classes $[\varphi]$ of coupling functionals φ: $X \times W \to \bar{R}$, such that each equivalence class $[\varphi]$ contains a unique representative φ_1 of type $\{0, -\infty\}$, which we shall denote by $(\varphi_1)_{[\varphi]}$; clearly, $(\varphi_1)_{[\varphi]} = (\varphi_1)_{\Delta_\varphi}$ and $[(\varphi_1)_{[\varphi]}] = [\varphi]$. If $\Delta \to [\varphi]$, we shall denote $[\varphi]$ by $[\varphi]_\Delta$ and Δ by $\Delta_{[\varphi]}$. Thus, $\Delta_{[\varphi]} = \Delta_{\varphi'}$, for any $\varphi' \in [\varphi]$.

Let us show now a relation between the hull operators associated to dualities and to coupling functionals. We recall that for any mapping Δ: $2^X \to 2^W$ *the dual mapping* Δ^*: $2^W \to 2^X$ is defined [2] by

$$\Delta^*(Q) = \bigcup_{\substack{B \subset X \\ Q \subset \Delta(B)}} B \qquad (Q \subset W) \qquad (2.14)$$

and *the hull operator associated to a duality* $\Delta: 2^X \to 2^W$ is, by definition [2, p. 25], the operator $\Delta^*\Delta: 2^X \to 2^X$. In particular, for any coupling functional $\varphi: X \times W \to \bar{R}$, it is natural to consider *the hull operator* $\Delta_\varphi^*\Delta_\varphi$: $2^X \to 2^X$, where $\Delta_\varphi: 2^X \to 2^W$ is the duality associated to φ; thus, by (2.1) and (2.14),

$$\Delta_\varphi^*\Delta_\varphi(A) = \{x \in X \mid \varphi(x, w) \geqslant -1 \ (w \in W, \inf_{y \in A} \varphi(y, w) \geqslant -1)\}, \tag{2.15}$$

for any $A \subset X$. In particular, by $\inf \emptyset = +\infty$, we have $\Delta_\varphi^*\Delta_\varphi(\emptyset) = \{x \in X \mid \inf_{w \in W} \varphi(x, w) \geqslant -1\}$; thus, $\Delta_\varphi^*\Delta_\varphi(\emptyset) = \emptyset$ if and only if

$$\inf_{w \in W} \varphi(x, w) < -1 \qquad (x \in X). \tag{2.16}$$

Also, it is obvious that if $\varphi \sim \varphi'$, then $\Delta_\varphi^*\Delta_\varphi = \Delta_{\varphi'}^*\Delta_{\varphi'}$.

On the other hand, generalizing the concept of "W-convex hull" of a set $A \subset X$, where $W \subset \bar{R}^X$, due to Fan [3] (the $(-W)$-convex hull of A, in the sense of [16, Definition 1.4]), let us give

DEFINITION 2.4. For any coupling functional $\varphi: X \times W \to \bar{R}$ we define *the hull operator* $H_\varphi: 2^X \to 2^X$ *associated to* φ by

$$H_\varphi(A) = \{x \in X \mid \varphi(x, w) \geqslant -d \ ((w, d) \in W \times R, \inf_{y \in A} \varphi(y, w) \geqslant -d)\}, \tag{2.17}$$

for any $A \subset X$. In particular, by $\inf \emptyset = +\infty$, we have $H_\varphi(\emptyset) = \{x \in X \mid \varphi(x, w) = +\infty \ (w \in W)\}$; thus, $H_\varphi(\emptyset) = \emptyset$ if and only if

$$\inf_{w \in W} \varphi(x, w) < +\infty \qquad (x \in X). \tag{2.18}$$

Obviously, we have

$$H_\varphi(A) \subset \Delta_\varphi^*\Delta_\varphi(A) \qquad (A \subset X), \tag{2.19}$$

and there arises the problem of finding conditions under which $H_\varphi = \Delta_\varphi^*\Delta_\varphi$. For simplicity, we shall consider only the case when $W \subset \bar{R}^X$ and $\varphi = \varphi_W$, the "natural coupling functional," i.e.,

$$\varphi_W(x, w) = w(x) \qquad (x \in X, w \in W); \tag{2.20}$$

note that, when $W \subset R^X$, φ_W is linear in w, but not necessarily bilinear.

THEOREM 2.2. *Let* $W \subset \bar{R}^X$ *be such that* $W + R \subset W$ *and define* φ_W *by* (2.20). *Then*

$$H_{\varphi_W}(A) = \Delta_{\varphi_W}^*\Delta_{\varphi_W}(A) \qquad (A \subset X). \tag{2.21}$$

Proof. By (2.19), we have to prove only the inclusion $\supset$ in (2.21). Let $A \subset X$ and $x \notin H_{\varphi W}(A)$, so there exist $w_0 \in W$ and $d_0 \in R$ with $\inf w_0(A) \geqslant -d_0$, such that $w_0(x) < -d_0$. Then, for $w' = w_0 + d_0 - 1 \in W + R \subset W$, we have $\inf w'(A) \geqslant -1$, $w'(x) < -1$, so $x \notin \Delta^*_{\varphi W} \Delta_{\varphi W}(A)$, which completes the proof.

Following Remark 1.1(b), let us consider, finally, the corresponding notions and results for polarities. By Definition 2.1 and Theorem 1.1, it is natural to give

DEFINITION 2.5. For any coupling functional $\varphi: X \times W \to \bar{R}$, we define *the set* $\Omega_\varphi \subset X \times W$ and *the polarity* $\rho(\Omega_\varphi): 2^X \to 2^W$ *associated to* φ, by

$$\Omega_\varphi = \{(x, w) \in X \times W \mid \varphi(x, w) \geqslant -1\}, \tag{2.22}$$

$$A^{\rho(\Omega_\varphi)} = \{w \in W \mid \varphi(x, w) \geqslant -1 \ (x \in A)\} \qquad (A \subset X); \tag{2.23}$$

note that, by (2.23) and (2.1), we have $\rho(\Omega_\varphi) = \Delta_\varphi$.

EXAMPLE 2.4. Let X be a locally convex space and $W = X^*$ and define $\varphi_i: X \times W \to R$ $(i = 0, 2, 3)$ by (2.3) and

$$\varphi_2(x, \Phi) = \Phi(x) - 2 \qquad (x \in X, \Phi \in X^*), \tag{2.24}$$

$$\varphi_3(x, \Phi) = \Phi(x) - 1 \qquad (x \in X, \Phi \in X^*); \tag{2.25}$$

note that these φ_i's are not of type $\{0, -\infty\}$. Then, by (2.23), we obtain, respectively,

$$A^{\rho(\Omega_{\varphi_0})} = \{\Phi \in X^* \mid \Phi(x) \geqslant -1 \ (x \in A)\} \qquad (A \subset X), \tag{2.26}$$

$$A^{\rho(\Omega_{\varphi_2})} = \{\Phi \in X^* \mid \Phi(x) \geqslant 1 \ (x \in A)\} \qquad (A \subset X), \tag{2.27}$$

$$A^{\rho(\Omega_{\varphi_3})} = \{\Phi \in X^* \mid \Phi(x) \geqslant 0 \ (x \in A)\} \qquad (A \subset X), \tag{2.28}$$

so $-A^{\rho(\Omega_{\varphi_i})}$ $(i = 0, 2, 3)$ are polar sets of A, used by various authors (see [4, Examples 3.16 (1), (2), (3) and the references therein]). Note that $-A^{\rho(\Omega_{\varphi_3})} = A^{\rho(\Omega)}$, where $\Omega = \{(x, \Phi) \mid (x, -\Phi) \in \Omega_{\varphi_3}\}$ and that the polarity $A \to -A^{\rho(\Omega_{\varphi_3})}$ is the case $\varphi = \varphi_0$ of the duality Δ^0_φ of (2.11); thus, Δ^0_φ coincides with the natural extension of the polarity $A \to -A^{\rho(\Omega_{\varphi_3})}$ to an arbitrary φ.

From Theorems 1.1 and 2.1 there follows

THEOREM 2.3. *For every set* $\Omega \subset X \times W$ (*or, equivalently, for every polarity* $\rho(\Omega): 2^X \to 2^W$) *there exists a unique coupling functional* φ_1 *of type*

$\{0, -\infty\}$, *such that* $\Omega = \Omega_{\varphi_1}$ (*or, equivalently, such that* $\rho(\Omega) = \rho(\Omega_{\varphi_1})$), *namely,*

$$\varphi_1(x, w) = -\chi_\Omega(x, w) = -\chi_{\{x\}^{\rho(\Omega)}}(w) \qquad (x \in X, w \in W). \tag{2.30}$$

Remark 2.6. Martínez-Legaz [7] has considered, for each set $\Omega \subset X \times W$, the coupling functional $\varphi = \chi_\Omega$ and the sets $\Gamma(x) = \{w \in W \mid (x, w) \in \Omega\}$ $(x \in X)$ (which are nothing else than $\{x\}^{\rho(\Omega)}$ above), in view of an application to conjugation (see Remark 4.3(c) below); however, he has not considered the converse direction (i.e., the set Ω_φ of (2.22)), nor the polar sets $A^{\rho(\Omega)}$ of (1.3).

3. Conjugations and Coupling Functionals

Let us recall that an operator $c: f \in \bar{R}^X \to f^c \in \bar{R}^W$ is called [15] a *conjugation,* if for every index set $I \neq \varnothing$ we have

$$(\inf_{i \in I} f_i)^c = \sup_{i \in I} f_i^c \qquad (\{f_i\}_{i \in I} \subset \bar{R}^X), \tag{3.1}$$

$$(f \dot{+} d)^c = f^c \dotplus -d \quad (f \in \bar{R}^X, d \in \bar{R}). \tag{3.2}$$

For example, if $\varphi: X \times W \to \bar{R}$ is any coupling functional, then the operator $c(\varphi): \bar{R}^X \to \bar{R}^W$, defined by

$$f^{c(\varphi)}(w) = \sup_{x \in X} \{\varphi(x, w) \dotplus -f(x)\} \qquad (f \in \bar{R}^X, w \in W), \tag{3.3}$$

is a conjugation. The functional $f^{c(\varphi)}$ of (3.3) is the well-known *generalized Fenchel conjugate of f with respect to the coupling functional* φ, introduced by Moreau [9, 10]. For the conjugations with respect to the coupling functionals φ of Examples 2.1–2.3 above, see [15, Examples 2.1–2.4]. In the opposite direction, according to [15, Theorem 3.1], *for every conjugation c*: $\bar{R}^X \to \bar{R}^W$ *there exists a unique coupling functional* φ_c: $X \times W \to \bar{R}$ *such that*

$$f^c(w) = \sup_{x \in X} \{\varphi_c(x, w) \dotplus -f(x)\} \qquad (f \in \bar{R}^X, w \in W) \tag{3.4}$$

(i.e., such that for each $f \in \bar{R}^X$ we have $f^c = f^{c(\varphi_c)}$, the generalized Fenchel conjugate of f with respect to φ_c), *namely,*

$$\varphi_c(x, w) = (\chi_{\{x\}})^c(w) \qquad (x \in X, w \in W); \tag{3.5}$$

φ_c is called [15] *the coupling functional associated to the conjugation c.* Thus, we have a *one-to-one correspondence between conjugations*

$c: \bar{R}^X \to \bar{R}^W$ *and coupling functionals* $\varphi: X \times W \to \bar{R}$. Also, for any conjugation $f \to f^c$ and any coupling functional φ, we have

$$c(\varphi_c) = c, \qquad \varphi_{c(\varphi)} = \varphi, \tag{3.6}$$

and each statement on conjugations has an equivalent counterpart on coupling functionals and vice versa; often we shall give both of them.

The relation of equivalence of coupling functionals, introduced in Definition 2.3, and the one-to-one correspondence between conjugations and coupling functionals, mentioned above, induce a relation of equivalence for conjugations, as follows:

DEFINITION 3.1. We shall say that two conjugations $c, c_1: \bar{R}^X \to \bar{R}^W$ are *equivalent*, and we shall write $c \sim c_1$, if for the associated coupling functionals $\varphi_c, \varphi_{c_1}: X \times W \to \bar{R}$ we have $\varphi_c \sim \varphi_{c_1}$, or, equivalently (by Remark 2.4),

$$\{w \in W \mid \varphi_c(x, w) \geqslant -1\} = \{w \in W \mid \varphi_{c_1}(x, w) \geqslant -1\} \qquad (x \in X). \tag{3.7}$$

Thus, for the corresponding equivalence classes we have, by (3.6),

$$[c] = \{c_1 \mid \varphi_{c_1} \sim \varphi_c\} = \{c(\varphi') \mid \varphi' \sim \varphi_c\}, \; [c(\varphi)] = \{c(\varphi') \mid \varphi' \sim \varphi\}. \tag{3.8}$$

From Corollary 2.1 and (3.6) we obtain

PROPOSITION 3.1. (a) *Each conjugation* $c: \bar{R}^X \to \bar{R}^W$ *is equivalent to a unique conjugation* $c(\varphi_1): \bar{R}^X \to \bar{R}^W$, *with* $\varphi_1: X \times W \to \bar{R}$ *of type* $\{0, -\infty\}$, *namely,* φ_1 *of* (2.13) *for* $\varphi = \varphi_c$ (*i.e.,* $\varphi_1 = (\varphi_1)_{[\varphi_c]}$ *of Remark* 2.5).

(b) *For any coupling functional* $\varphi: X \times W \to \bar{R}$, *the conjugation* $c(\varphi)$ *of* (3.3) *is equivalent to a unique conjugation* $c(\varphi_1)$, *with* $\varphi_1: X \times W \to \bar{R}$ *of type* $\{0, -\infty\}$, *namely,* $\varphi_1 = (\varphi_1)_{[\varphi]}$ (*of* (2.13) *and Remark* 2.5).

Remark 3.1. By the above, *each equivalence class* $[c]$ *of conjugations contains a unique representative* $c(\varphi_1)$ *with* φ_1 *of type* $\{0, -\infty\}$.

Now we shall obtain explicit formulae for $c(\varphi_1)$. First, for the conjugations $c(\varphi_1)$ of Proposition 3.1(b), we prove

THEOREM 3.1. *For any coupling functional* $\varphi: X \times W \to \bar{R}$, *the conjugation* $c(\varphi_1): \bar{R}^X \to \bar{R}^W$, *where* $\varphi_1 = (\varphi_1)_{[\varphi]}$ (*the unique coupling functional of type* $\{0, -\infty\}$, *equivalent to* φ), *coincides with the operator* $L(\varphi): \bar{R}^X \to \bar{R}^W$, *where*

$$f^{L(\varphi)}(w) = -\inf_{\substack{x \in X \\ \varphi(x,w) \geqslant -1}} f(x) \qquad (f \in \bar{R}^X, w \in W). \tag{3.9}$$

Proof. By (3.3), $\varphi_1(X\times W)\subset\{0, -\infty\}$, (2.13), and (3.9), we have, for all $f\in\bar{R}^X$ and $w\in W$,

$$f^{c(\varphi_1)}(w)=\sup_{x\in X}\{\varphi_1(x,w)\dot{+}-f(x)\}=\sup_{\substack{x\in X\\ \varphi_1(x,w)=0}}\{0\dot{+}-f(x)\}$$

$$=\sup_{\substack{x\in X\\ \varphi(x,w)\geqslant -1}}\{0\dot{+}-f(x)\}=-\inf_{\substack{x\in X\\ \varphi(x,w)\geqslant -1}} f(x)=f^{L(\varphi)}(w).$$

COROLLARY 3.1. *If $\varphi_1: X\times W\to\bar{R}$ is of type $\{0, -\infty\}$, then*

$$f^{c(\varphi_1)}(w)=f^{L(\varphi_1)}(w)=-\inf_{\substack{x\in X\\ \varphi_1(x,w)\geqslant -1}} f(x)\qquad (f\in\bar{R}^X,\ w\in W).\quad (3.10)$$

It will be useful to look at Theorem 3.1 also in the converse direction. Namely, Theorem 3.1 shows, in particular, that for any coupling functional $\varphi: X\times W\to\bar{R}$ (not necessarily of type $\{0, -\infty\}$), the operator $L(\varphi): \bar{R}^X\to\bar{R}^W$, defined by (3.9), is a conjugation; of course, this follows also directly, verifying (3.1) and (3.2).

DEFINITION 3.2. For any coupling functional $\varphi: X\times W\to\bar{R}$, we call the conjugation $L(\varphi): \bar{R}^X\to\bar{R}^W$ defined by (3.9) *the conjugation of type Lau associated to φ.*

EXAMPLE 3.1. Let X be a locally convex space, $W=X^*$ and $\varphi=\varphi_0$ of (2.3). Then the conjugation of type Lau $L(\varphi_0): \bar{R}^X\to\bar{R}^{X^*}$ associated to φ_0 is

$$f\to f^{L(\varphi_0)}(\Phi)=-\inf_{\substack{x\in X\\ \Phi(x)\geqslant -1}} f(x)\qquad (\Phi\in X^*).\quad (3.11)$$

Remark 3.2. (a) It is exactly because of the form (3.3) of $c(\varphi)$ and the need of computing it for the coupling functionals $\varphi=\varphi_1$ of a Theorem 2.1-type result, that *we have to work with φ_1's of type $\{0, -\infty\}$ and to define $\Delta_\varphi(A)$ and Ω_φ by* (2.1) *and* (2.22), *respectively.* Indeed, if we change the definition of $\Delta_\varphi(A)$ as suggested in Remark 2.2, then we have to replace φ_1 of Theorem 2.1, formula (2.9), by $\varphi_1(x,w)=\chi_{\Delta(\{x\})}(w)$, which is "of type $\{0, +\infty\}$." However, for such a $\varphi=\varphi_1$, formula (3.3) yields $f^c(w)=+\infty$ whenever there exists $x_0\in X$ with $\varphi_1(x_0,w)=+\infty$ and $f(x_0)<+\infty$. One could remedy this deficiency, *replacing the generalized Fenchel conjugate* (3.3) *of f with respect to φ by the "lower φ-conjugate" of f in the sense of Lindberg* [6], defined by

$$f^{\cap}(w)=f^{\cap(\varphi)}(w)=\inf_{x\in X}\{\varphi(x,w)\dot{+}f(x)\}\qquad (f\in\bar{R}^X,\ w\in W);\ (3.12)$$

indeed, although $\cap: \bar{R}^X \to \bar{R}^W$ is not a "conjugation" in the sense of (3.1), (3.2), one can show that it satisfies, instead,

$$(\inf_{i \in I} f_i)^{\cap} = \inf_{i \in I} f_i^{\cap} \qquad (\{f_i\}_{i \in I} \subset \bar{R}^X), \tag{3.13}$$

$$(f \dotplus d)^{\cap} = f^{\cap} \dotplus d \quad (f \in \bar{R}^X, d \in \bar{R}), \tag{3.14}$$

and that, conversely, for every operator $\cap: \bar{R}^X \to \bar{R}^W$ satisfying (3.13), (3.14), there exists a unique coupling functional $\varphi: X \times W \to \bar{R}$ such that (3.12) holds, namely, $\varphi = \varphi_c$ of (3.5). Applying (3.12), in particular, to φ "of type $\{0, +\infty\}$," we obtain, similarly to (3.9), (3.10),

$$f^{\cap(\varphi)}(w) = \inf_{\substack{x \in X \\ \varphi(x,w) \leqslant 1}} f(x) \qquad (f \in \bar{R}^X, w \in W). \tag{3.15}$$

The operator $\cap(\varphi): \bar{R}^X \to \bar{R}^W$ of (3.15) may be also considered *for an arbitrary coupling functional* $\varphi: X \times W \to \bar{R}$, and may be called "of type Lau." Indeed, for a locally convex space X, $W = X^*$ and $\varphi = \varphi_0$ of (2.3), formula (3.15) becomes

$$f^{\cap(\varphi_0)}(\Phi) = \inf_{\substack{x \in X \\ \Phi(x) \leqslant 1}} f(x) = -f^{L(\varphi_0)}(-\Phi) = -f^{L(-\varphi_0)}(\Phi)$$

$$(f \in \bar{R}^X, \Phi \in X^*); \tag{3.16}$$

in the particular case when $X = R^n$, the operator $\cap(\varphi_0): \bar{R}^X \to \bar{R}^{X^*}$, defined by (3.16), has been considered by Lau [5] (see also [1, Chap. VI, Sect. 4]). Let us also observe that for φ of type $\{0, +\infty\}$ we have

$$\{x \in X \mid \varphi(x, w) \leqslant 1\} = \{x \in X \mid \varphi(x, w) \leqslant 0\} \qquad (w \in W), \tag{3.17}$$

and thus, for any such φ, (3.15) coincides with

$$f^{\cap(\varphi)}(w) = \inf_{\substack{x \in X \\ \varphi(x,w) \leqslant 0}} f(x) \qquad (f \in \bar{R}^X, w \in W). \tag{3.18}$$

The "conjugation" $\cap(\varphi): \bar{R}^X \to \bar{R}^W$, with $f^{\cap(\varphi)}$ of (3.18) and *an arbitrary* coupling functional φ, is related to the duality Δ_φ^0 of (2.11) (see also Example 2.4). This "conjugation" $\cap(\varphi)$ has been studied, for $\varphi = \varphi_0$ of (2.3), by Oettli [11]; for (3.18) with $\varphi = \varphi_0$ and $\leqslant$ replaced by $\geqslant$, see Martínez Legaz [8].

For the "conjugations" $f \to f^{\cap}$ of (3.13), (3.14), *and* $f \to f^{\cap(\varphi)}$ *of* (3.15) *one can prove results corresponding to those for* $f \to f^c$ *and* $f \to f^{c(\varphi)}$, which we leave to the reader. In the sequel we shall continue to use conjugations in the sense (3.1), (3.2), and the (generalized Fenchel) conjugate (3.3).

(b) For φ_1 of type $\{0, -\infty\}$, we have

$$\{x \in X \mid \varphi_1(x, w) \geqslant -1\} = \{x \in X \mid \varphi_1(x, w) \geqslant 0\} \qquad (w \in W), \quad (3.19)$$

and thus, for any such φ_1 (3.10) coincides with

$$f^{c(\varphi_1)}(w) = -\inf_{\substack{x \in X \\ \varphi_1(x,w) \geqslant 0}} f(x) \qquad (f \in \bar{R}^X, w \in W). \quad (3.20)$$

Again, one can consider, for an arbitrary coupling functional $\varphi: X \times W \to \bar{R}$, the operator $*(\varphi): \bar{R}^X \to \bar{R}^W$, defined by

$$f^{*(\varphi)}(w) = -\inf_{\substack{x \in X \\ \varphi(x,w) \geqslant 0}} f(x) \qquad (f \in \bar{R}^X, w \in W) \quad (3.21)$$

and it turns out, similarly to the proof of Theorem 3.1 (replacing φ_1 of (2.13) by $\varphi_1'(x, w) = 0$ if $\varphi(x, w) \geqslant 0$ and $-\infty$ if $\varphi(x, w) < 0$, which amounts to replacing $\geqslant -1$ in the Definition (2.1) of Δ_φ by $\geqslant 0$ and working with the corresponding new equivalence classes of coupling functionals), that $*(\varphi)$ is a conjugation; of course, this follows also directly, verifying (3.1) and (3.2). For $\varphi = \varphi_0$ of (2.3), this conjugation $*(\varphi)$ has been used by Passy and Prisman [12].

Continuing to look at Theorem 3.1 in the converse direction, let us give

COROLLARY 3.2. *For any conjugation of type Lau $L(\varphi): \bar{R}^X \to \bar{R}^W$, there exists a unique coupling functional φ_1 of type $\{0, -\infty\}$, such that $L(\varphi) = c(\varphi_1)$ (or, equivalently, such that $L(\varphi) = L(\varphi_1)$), namely, $\varphi_1 = (\varphi_1)_{[\varphi]}$.*

Proof. By Theorem 3.1 and Corollary 3.1, for $\varphi_1 = (\varphi_1)_{[\varphi]}$ we have $L(\varphi) = c(\varphi_1) = L(\varphi_1)$. On the other hand, if φ_1, φ_1' are coupling functionals of type $\{0, -\infty\}$, such that $c(\varphi_1) = c(\varphi_1')$, or, equivalently (by Corollary 3.1), such that $L(\varphi_1) = L(\varphi_1')$, then, by the one-to-one correspondence between coupling functionals and conjugations, we obtain $\varphi_1 = \varphi_1'$.

COROLLARY 3.3. *For two coupling functionals $\varphi, \varphi': X \times W \to \bar{R}$, we have $L(\varphi) = L(\varphi')$ if and only if $\varphi \sim \varphi'$.*

Proof. Let $\varphi_1 = (\varphi_1)_{[\varphi]}$, $\varphi_1' = (\varphi_1')_{[\varphi']}$. Then, by Corollary 3.2, we have $L(\varphi) = L(\varphi')$ if and only if $c(\varphi_1) = c(\varphi_1')$, which, by the one-to-one correspondence between coupling functionals and conjugations, holds if and only if $\varphi_1 = \varphi_1'$. But, by Corollary 2.1, the latter equality holds if and only if $\varphi \sim \varphi'$.

Remark 3.3. (a) Conversely, Corollary 3.2 follows from Corollaries 3.1 and 3.3, applied to $\varphi' = \varphi_1 = (\varphi_1)_{[\varphi]}$ (for the existence part) and Corollary 2.1 (for the uniqueness part).

(b) Let us also mention the following direct proof of Corollary 3.3: If $\varphi \sim \varphi'$, then, by (2.12′) and (3.9), we have $L(\varphi) = L(\varphi')$. For the converse part, observe that by (3.9) for $f = \chi_{\{x\}}$ we have

$$(\chi_{\{x\}})^{L(\varphi)}(w) = -\inf_{\substack{y \in X \\ \varphi(y,w) \geqslant -1}} \chi_{\{x\}}(y) = -\chi_{\{y \in X \mid \varphi(y,w) \geqslant -1\}}(x), \quad (3.22)$$

for any $x \in X$ and $w \in W$. Thus, if $L(\varphi) = L(\varphi')$, then (2.12′) holds and hence, by Remark 2.4, $\varphi \sim \varphi'$.

(c) By the above, *we have a one-to-one correspondence between equivalence classes of coupling functionals* $[\varphi]$ *and conjugations of type Lau* $L(\varphi')$, *where* φ' *is any fixed representative of* $[\varphi]$ (in particular, we may choose $\varphi' = (\varphi_1)_{[\varphi]}$); therefore, we can introduce the notation

$$L([\varphi]) = L(\varphi'), \quad (3.23)$$

with φ' as above.

(d) *For any coupling functional* $\varphi: X \times W \to \bar{R}$, *the equivalence class of conjugations* $[c(\varphi)]$ *contains a unique conjugation of type Lau, namely,* $L(\varphi)$ (which coincides with $L(\varphi_1) = c(\varphi_1)$ of Corollaries 3.2, 3.1, and Proposition 3.1(b)).

Remark 3.4. (a) Using (3.6) and the above results, one obtains their equivalent counterparts for conjugations. Thus (see Theorem 3.1), *for any conjugation* $c: \bar{R}^X \to \bar{R}^W$, *the unique representative* $c(\varphi_1)$ *of* $[c]$, *provided by Proposition* 3.1(a), *coincides with the conjugation of type Lau* $L(\varphi_c)$, *where* φ_c *is the coupling functional* (3.5) *associated to the conjugation* c. Also (see Corollary 3.3), *for two conjugations* $c, c_1: \bar{R}^X \to \bar{R}^W$ *we have* $c \sim c_1$ *if and only if* $L(\varphi_c) = L(\varphi_{c_1})$. Furthermore (Remark 3.3(d)), *each equivalence class* $[c]$ *of conjugations contains a unique representative of type Lau, namely,* $L(\varphi_c)$. Therefore, one can also denote $L(\varphi_c)$ by $L(c)$ or $L([c])$; then, by (3.6), $L(\varphi) = L(c(\varphi)) = L([c(\varphi)])$.

(b) Similarly to Section 2, one can introduce notations such as $(\varphi_1)_c$, $(\varphi_1)_{[c]}$, $(\varphi_1)_{L(\varphi)}$, etc., with obvious meanings, and one can observe some relations, such as $(\varphi_1)_{[c]} = (\varphi_1)_{[\varphi_c]}$, $(\varphi_1)_{L(\varphi)} = (\varphi_1)_{[L(\varphi)]} = (\varphi_1)_{[c(\varphi)]} = (\varphi_1)_{[\varphi]}$, etc., which we leave to the reader.

(c) A result for functionals, corresponding to Theorem 2.2, namely, a connection between hull operators $f \in \bar{R}^X \to f^{cc^*} \in \bar{R}^X$, associated to conjugations $c: \bar{R}^X \to \bar{R}^W$ (where $c^*: \bar{R}^W \to \bar{R}^X$ is a suitably defined "dual" of c) and hull operators associated to coupling functionals $\varphi: X \times W \to \bar{R}$, has been given in [15, Theorem 5.1] (see also [14, Remark 5.6(a); 16, Theorem 4.1]).

4. Conjugations and Dualities

Definition 4.1. For any conjugation $c: \bar{R}^X \to \bar{R}^W$ we define *the duality* $\Delta_c: 2^X \to 2^W$ *associated to* c, by

$$\Delta_c(A) = \{w \in W \mid (\chi_{\{x\}})^c(w) \geqslant -1 \ (x \in A)\} \qquad (A \subset X). \tag{4.1}$$

Proposition 4.1. (a) *For every conjugation* $c: \bar{R}^X \to \bar{R}^W$ *we have*

$$\Delta_c = \Delta_{\varphi_c}, \tag{4.2}$$

where $\varphi_c: X \times W \to \bar{R}$ *is the coupling functional* (3.5) *associated to* c *and* Δ_{φ_c} *is the duality* (2.1) (*with* $\varphi = \varphi_c$) *associated to* φ_c.

(b) *For every coupling functional* $\varphi: X \times W \to \bar{R}$ *we have*

$$\Delta_{c(\varphi)} = \Delta_\varphi. \tag{4.3}$$

Proof. (a) By (4.1), (3.5), and (2.1) with $\varphi = \varphi_c$, we have

$$\Delta_c(A) = \{w \in W \mid \varphi_c(x, w) \geqslant -1 \ (x \in A)\} = \Delta_{\varphi_c}(A) \qquad (A \subset X).$$

Finally (b) follows from (4.2) and (3.6).

Example 4.1. Let X be a locally convex space, $W = X^*$ and $c: \bar{R}^X \to \bar{R}^{X^*}$ the usual Fenchel conjugation, i.e.,

$$f^c(\Phi) = \sup_{x \in X} \{\Phi(x) - f(x)\} \qquad (f \in \bar{R}^X, \Phi \in X^*). \tag{4.4}$$

Then, by Proposition 4.1(a) and Example 2.1, we obtain

$$\Delta_c(A) = \Delta_{\varphi_0}(A) = \{\Phi \in X^* \mid \Phi(x) \geqslant -1 \ (x \in A)\} \qquad (A \subset X). \tag{4.5}$$

Corollary 4.1. *For two conjugations* $c, c_1: \bar{R}^X \to \bar{R}^W$, *we have* $c \sim c_1$ *if and only if* $\Delta_c = \Delta_{c_1}$.

Proof. By Definition 3.1 we have $c \sim c_1$ if and only if $\varphi_c \sim \varphi_{c_1}$, which, by (4.2) and Definition 2.3, is equivalent to $\Delta_c = \Delta_{c_1}$.

Remark 4.1. (a) By the above, *we have a one-to-one correspondence between equivalence classes* $[c]$ *of conjugations* $c: \bar{R}^X \to \bar{R}^W$ *and dualities* $\Delta: 2^X \to 2^W$. If $[c] \to \Delta$, we shall denote Δ by $\Delta_{[c]}$ and $[c]$ by $[c]_\Delta$. Thus, $\Delta_{[c]} = \Delta_{c_1}$, for any $c_1 \in [c]$.

(b) By Section 3, every conjugation of type Lau $L(\varphi)\colon \bar{R}^X \to \bar{R}^W$ (see Definition 3.2) is a conjugation (3.1), (3.2), and thus, by Definition 4.1, *the duality* $\Delta_{L(\varphi)}\colon 2^X \to 2^W$ *associated to* $L(\varphi)$ is

$$\Delta_{L(\varphi)}(A) = \{w \in W \mid (\chi_{\{x\}})^{L(\varphi)}(w) \geqslant -1 \ (x \in A)\} \qquad (A \subset X); \quad (4.6)$$

also, by Corollary 3.2, we have

$$\Delta_{L(\varphi)} = \Delta_{c(\varphi_1)}, \tag{4.7}$$

where $\varphi_1 = (\varphi_1)_{[\varphi]}$, the unique coupling functional of type $\{0, -\infty\}$, equivalent to φ.

PROPOSITION 4.2. *For every conjugation of type Lau* $L(\varphi)\colon \bar{R}^X \to \bar{R}^W$, *there holds*

$$\Delta_{L(\varphi)} = \Delta_{\varphi}. \tag{4.8}$$

Proof. By Remark 3.3(d), we have $L(\varphi) \sim c(\varphi)$, whence, by Corollary 4.1 and formula (4.3), we obtain

$$\Delta_{L(\varphi)} = \Delta_{c(\varphi)} = \Delta_{\varphi};$$

or, alternatively, by (4.7), (4.3), and $\varphi_1 \sim \varphi$, we get

$$\Delta_{L(\varphi)} = \Delta_{c(\varphi_1)} = \Delta_{\varphi_1} = \Delta_{\varphi};$$

or, alternatively, by (4.6), (3.22), and (2.1), we have

$$\begin{aligned}\Delta_{L(\varphi)}(A) &= \{w \in W \mid (\chi_{\{x\}})^{L(\varphi)}(w) \geqslant -1 \ (x \in A)\} \\ &= \{w \in W \mid -\chi_{\{y \in X \mid \varphi(y,w) \geqslant -1\}}(x) \geqslant -1 \ (x \in A)\} \\ &= \{w \in W \mid \varphi(x, w) \geqslant -1 \ (x \in A)\} = \Delta_{\varphi}(A) \qquad (A \subset X).\end{aligned}$$

In the opposite direction, let us prove

THEOREM 4.1. *For every duality* $\Delta\colon 2^X \to 2^W$ *there exists a unique conjugation of type Lau* $L(\Delta) = L(\varphi)\colon \bar{R}^X \to \bar{R}^W$, *such that* $\Delta = \Delta_{L(\varphi)}$, *namely,*

$$f \to f^{L(\Delta)}(w) = -\inf_{\substack{x \in X \\ w \in \Delta(\{x\})}} f(x) \qquad (w \in W). \tag{4.9}$$

Proof. By Theorem 2.1 and Proposition 4.2, for the given duality Δ there exists a unique coupling functional φ of type $\{0, -\infty\}$, such that $\Delta = \Delta_{\varphi} = \Delta_{L(\varphi)}$. By (2.1), we have $\varphi(x, w) \geqslant -1$ if and only if $w \in \Delta_{\varphi}(\{x\}) = \Delta(\{x\})$, whence, by (3.9) and $L(\Delta) = L(\varphi)$, we obtain (4.9).

Remark 4.2. (a) Theorem 4.1 shows also that for any duality $\Delta: 2^X \to 2^W$, the mapping $L(\Delta): \bar{R}^X \to \bar{R}^W$, defined by (4.9), is a conjugation of type Lau; we shall call it *the conjugation of type Lau associated to the duality* Δ. For $\Delta = \Delta_\varphi$ of (2.1), formula (4.9) yields again (3.9).

(b) By (2.14) we have $w \in \Delta(\{x\})$ if and only if $x \in \Delta^*(\{w\})$, so we can write (4.9) also in the form

$$f \to f^{L(\Delta)}(w) = -\inf_{x \in \Delta^*(\{w\})} f(x) \qquad (w \in W). \tag{4.10}$$

(c) By Theorem 4.1 and Section 3, we have one-to-one correspondences between dualities $\Delta: 2^X \to 2^W$, conjugations of type Lau $L(\Delta)$: $\bar{R}^X \to \bar{R}^W$, and equivalence classes $[c]$ of conjugations $c: \bar{R}^X \to \bar{R}^W$. The equivalence class (in the sense of Definition 3.1) of the conjugation of type Lau $L(\Delta)$ associated to the duality Δ coincides with $[c]_\Delta$ of Remark 4.1 (by Remark 3.3(d)).

Finally, following Remark 1.1(b), let us mention, briefly, the corresponding notions and results for polarities $\rho(\Omega): 2^X \to 2^W$, or, equivalently, for subsets Ω of $X \times W$.

Definition 4.2. For any conjugation $c: \bar{R}^X \to \bar{R}^W$, we define *the set* $\Omega_c \subset X \times W$ and *the polarity* $\rho(\Omega_c): 2^X \to 2^W$ *associated to* c, by

$$\Omega_c = \{(x, w) \in X \times W \mid (\chi_{\{x\}})^c(w) \geqslant -1\}, \tag{4.11}$$

$$A^{\rho(\Omega_c)} = \{w \in W \mid (\chi_{\{x\}})^c(w) \geqslant -1 \ (x \in A)\} \qquad (A \subset X); \tag{4.12}$$

note that, by (4.12) and (4.1), we have $\rho(\Omega_c) = \Delta_c$.

Similarly to Proposition 4.1, there holds

Proposition 4.3. (a) *For every conjugation* $c: \bar{R}^X \to \bar{R}^W$ *we have*

$$\Omega_c = \Omega_{\varphi_c}, \tag{4.13}$$

where Ω_{φ_c} *is the set* (2.22) *with* $\varphi = \varphi_c$.

(b) *For every coupling functional* $\varphi: X \times W \to \bar{R}$ *we have*

$$\Omega_{c(\varphi)} = \Omega_\varphi. \tag{4.14}$$

Similarly to Remark 4.1(b), for any conjugation of type Lau $L(\varphi)$: $\bar{R}^X \to \bar{R}^W$, *the set* $\Omega_{L(\varphi)} \subset X \times W$ and *the polarity* $\rho(\Omega_{L(\varphi)}): 2^X \to 2^W$ *associated to* $L(\varphi)$, satisfy

$$\Omega_{L(\varphi)} = \Omega_{c(\varphi_1)}, \qquad \rho(\Omega_{L(\varphi)}) = \rho(\Omega_{c(\varphi_1)}), \tag{4.15}$$

where $\varphi_1 = (\varphi_1)_{[\varphi]}$.

Similarly to Proposition 4.2, there holds

PROPOSITION 4.4. *For every conjugation of type Lau* $L(\varphi)\colon \bar{R}^X \to \bar{R}^W$, *we have*

$$\Omega_{L(\varphi)} = \Omega_\varphi, \qquad \rho(\Omega_{L(\varphi)}) = \rho(\Omega_\varphi). \tag{4.16}$$

Finally, from Theorems 1.1 and 4.1 there follows

THEOREM 4.2. *For every set* $\Omega \subset X \times W$ *(or, equivalently, for every polarity* $\rho(\Omega)\colon 2^X \to 2^W$*) there exists a unique conjugation of type Lau* $L(\Omega) = L(\varphi)\colon \bar{R}^X \to \bar{R}^W$, *such that* $\Omega = \Omega_{L(\varphi)}$ *(or, equivalently, such that* $\rho(\Omega) = \rho(\Omega_{L(\varphi)})$*), namely,*

$$f \to f^{L(\Omega)}(w) = -\inf_{\substack{x \in X \\ (x,w) \in \Omega}} f(x) \qquad (w \in W). \tag{4.17}$$

Remark 4.3. (a) Theorem 4.2 shows that for any set $\Omega \subset X \times W$, the mapping $L(\Omega)\colon \bar{R}^X \to \bar{R}^W$, defined by (4.17), is a conjugation of type Lau. For $\Omega = \Omega_\varphi$ and $\Omega = \Omega_\Delta$ of (2.22) and (1.6), formula (4.17) yields again (3.9) and (4.9), respectively.

(b) By Theorem 4.2 and Section 3, we have one-to-one correspondences between sets $\Omega \subset X \times W$ (or, equivalently, polarities $\rho(\Omega)\colon 2^X \to 2^W$), conjugations of type Lau $L(\Omega)\colon \bar{R}^X \to \bar{R}^W$, and equivalence classes $[c]$ of conjugations $c\colon \bar{R}^X \to \bar{R}^W$ (see also the other observations in Remark 4.2(a), (c)).

(c) Martínez Legaz [7] has observed that if $\Omega \subset X \times W$ nd $\varphi = \chi_\Omega$, then the lower φ-conjugate (3.12) of $f\colon X \to \bar{R}$ becomes

$$f^{\cap}(w) = f^{\cap(\Omega)}(w) = \inf_{\substack{x \in X \\ (x,w) \in \Omega}} f(x) \qquad (w \in W), \tag{4.18}$$

and has called $f \to f^{\cap(\Omega)}$ "the conjugation associated to Ω"; however, he has not considered the converse direction, i.e., the set $\Omega_\cap = \{(x, w) \in X \times W \mid (\chi_{\{x\}})^{\cap}(w) \leqslant 1\}$ (see also Remark 2.6 above).

5. FURTHER CONNECTIONS, VIA INDICATOR MAPPINGS AND LEVEL SET MAPPINGS

In order to express some relations between dualities $\Delta\colon 2^X \to 2^W$ and conjugations $\bar{R}^X \to \bar{R}^W$, it is also useful to consider the "*indicator mappings*" $\chi_\cdot\colon G \in 2^X \to \chi_G \in \bar{R}^X$ and the ("*upper*" and "*lower*") "*level set mappings*" $T_d\colon \bar{R}^X \to 2^X$ and $S_d\colon \bar{R}^X \to 2^X$, defined by

$$T_d(f) = \{x \in X \mid f(x) \geqslant d\} \qquad (f \in \bar{R}^X, d \in R), \tag{5.1}$$

$$S_d(f) = \{x \in X \mid f(x) \leqslant d\} \qquad (f \in \bar{R}^X, d \in R). \tag{5.2}$$

The mappings $\chi.$ and $-\chi.$ are one-to-one, while the mappings T_d and S_d are onto. Moreover, we have

$$T_{-d}(-\chi_A)=S_d(\chi_A)=A \qquad (A\subset X, d\geqslant 0), \tag{5.3}$$

i.e., for each $d\geqslant 0$, the diagram

$$\begin{tikzcd} 2^X \arrow[r, "-\chi."] \arrow[d, "\chi."'] \arrow[dr, "E"] & \bar{R}^X \arrow[d, "T_{-d}"] \\ \bar{R}^X \arrow[r, "S_d"] & 2^X \end{tikzcd} \tag{5.4}$$

is commutative (where E denotes the identical mapping); also ([16, Remark 1.11(a)]), we have

$$S_{-d}(-\chi_{X\setminus A})=A \qquad (A\subset X, d\geqslant 0). \tag{5.5}$$

Furthermore, let us mention that, for any $\{A_i\}_{i\in I}\subset 2^X$, $\{f_i\}_{i\in I}\subset \bar{R}^X$ and $d\in R$,

$$\chi_{\bigcup_{i\in I}A_i}=\inf_{i\in I}\chi_{A_i}, \qquad \chi_{\bigcap_{i\in I}A_i}=\sup_{i\in I}\chi_{A_i}, \tag{5.6}$$

$$T_d(\inf_{i\in I}f_i)=\bigcap_{i\in I}T_d(f_i), \qquad S_d(\sup_{i\in I}f_i)=\bigcap_{i\in I}S_d(f_i), \tag{5.7}$$

and, if $c\colon \bar{R}^X\to\bar{R}^W$ is a conjugation, then, by $A=\bigcup_{x\in A}\{x\}$, (5.6), and (3.1), we obtain

$$(\chi_A)^c=\sup_{x\in A}(\chi_{\{x\}})^c \qquad (A\subset X); \tag{5.8}$$

if $W\subset\bar{R}^X$ and $\varphi_c=\varphi_W$ of (2.20), then (5.8) is the "support functional" of the set A. Some relations between set hulls and functional hulls, via indicator mappings and lower level set mappings, have been given in [14, 16].

Here we want to make only a remark about the possibility of composing indicator mappings, level set mappings, dualities, and conjugations. By (4.9), we have

$$(\chi_{\{x\}})^{L(\Delta)}(w)=-\chi_{\Delta(\{x\})}(w) \qquad (x\in X, w\in W), \tag{5.9}$$

i.e., the diagram

$$
\begin{array}{ccc}
X & \xrightarrow{(\chi|x).} & \bar{R}^X \\
{\scriptstyle \Delta|X}\downarrow & & \downarrow{\scriptstyle L(\Delta)} \\
2^W & \xrightarrow{-\chi.} & \bar{R}^W
\end{array} \tag{5.10}
$$

is commutative, where X is identified with the family of all singletons $\{x\}$, $x \in X$; actually, the left-hand side of (5.9) is the unique coupling functional $(\varphi_1)_{L(\Delta)}$ of type $\{0, -\infty\}$ associated to $L(\Delta)$, while the right-hand side is $(\varphi_1)_\Delta$ of Remark 2.3(a). However, one cannot replace in (5.9) the singletons $\{x\}$ by arbitrary subsets $A \subset X$, since by (1.2), (5.6), (5.8), and (5.9), we have

$$-\chi_{\Delta(A)} = -\chi_{\bigcap_{x \in A} \Delta(\{x\})} = -\sup_{x \in A} \chi_{\Delta(\{x\})} = \inf_{x \in A} (-\chi_{\Delta(\{x\})}),$$

$$(\chi_A)^{L(\Delta)} = \sup_{x \in A} (\chi_{\{x\}})^{L(\Delta)} = \sup_{x \in A} (-\chi_{\Delta(\{x\})}).$$

A similar remark can be made also for (3.22) written in the form

$$(\chi_{\{x\}})^{L(\varphi)}(w) = -\chi_{\{w' \in W \mid \varphi(x, w') \geqslant -1\}}(w) \qquad (x \in X, w \in W), \tag{5.11}$$

using the upper level sets $T_{-1}(\varphi_x)$ of the "partial functionals" $\varphi_x \colon W \to \bar{R}$ $(x \in X)$ associated to φ, defined by

$$\varphi_x(w) = \varphi(x, w) \qquad (x \in X, w \in W); \tag{5.12}$$

namely, the diagram

$$
\begin{array}{ccc}
X & \xrightarrow{(\chi|x).} & \bar{R}^X \\
{\scriptstyle T_{-1}(\varphi.)}\downarrow & & \downarrow{\scriptstyle L(\varphi)} \\
2^W & \xrightarrow{-\chi.} & \bar{R}^W
\end{array} \tag{5.13}
$$

is commutative.

Addendum. Generalizing (3.16), *for any coupling functional* $\varphi \colon X \times W \to \bar{R}$ we have, by (3.3), (3.12), and [10, formula (2.1)],

$$
\begin{aligned}
f^{\cap(\varphi)}(w) = \inf_{x \in X} \{\varphi(x, w) \dot{+} f(x)\} &= -\sup_{x \in X} \{-\varphi(x, w) \underset{\cdot}{+} -f(x)\} \\
&= -f^{c(-\varphi)}(w) \qquad (w \in W),
\end{aligned} \tag{5.14}
$$

so *the theories of generalized Fenchel conjugation* $f \to f^{c(\varphi)}$ *and Lindberg conjugation* $f \to f^{\cap(\varphi)}$ *are equivalent.*

Note added in proof. See also M. Volle, Conjugaisons par tranches, *Ann. Math. Pura Appl.* **139** (1985), 279–311 (whose manuscript we received after this paper had gone to print) and I. Singer, Infimal generators and dualities between complete lattices, preprint, INCREST 47/July 1985.

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